

A COMPARATIVE ANALYSIS BETWEEN DISJUNCTIVE KRIGING AND ORDINARY KRIGING FOR ESTIMATING THE RESERVE OF A MINE; A CASE STUDY OF CHOGHART IRON ORE DEPOSIT

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Abstract

*From a geostatistical viewpoint, non-linear interpolation is an attempt to estimate the conditional expectation, and further the conditional distribution of grade at a location, as opposed to simply predicting the grade itself. There are many non-linear methods now available. Disjunctive Kriging is one of them. This paper presents a comparison of ordinary kriging and disjunctive kriging in Choghart iron ore deposit in Yazd province, Iran. The case study consists of borehole samples measuring the Fe concentration. The sample data used in this study consist of exploration drilling data, suitably composited. The data set is irregularly spaced and has an almost normal distribution. Fe grade was selected as the major regional variable on which the present research has focused. To carry out ordinary kriging and disjunctive kriging, spherical model was fitted over empirical variogram. To estimate the Iron grade, ordinary kriging and disjunctive kriging methods were used. All of the exploitable blocks with dimensions 20*20*12.5 (m³) were estimated. In the case of Choghart iron ore deposit the average of disjunctive kriging estimation error variance is 72.076 while the average of ordinary kriging estimation error variance is 100.278. So estimation with ordinary kriging is more risky.*

Key words: geostatistics; ordinary kriging; disjunctive kriging; estimation error variance; Choghart iron ore deposit.

1. Introduction

From a geostatistical point of view, non-linear interpolation is an attempt to estimate the conditional expectation, and further the conditional distribution of grade at a location, as opposed to simply predicting the grade itself [1]. Linear estimation of regionalized variables (for example by inverse distance weighting or ordinary Kriging) results in relatively high estimation variances, i.e. the estimates have very low precision. Assessment of project economics (or other critical decision making) based on linear

estimation is therefore risky. Non-linear estimation methods like disjunctive kriging perform better and the lower estimation variance allows less risky economic decision-making.

Another advantage of disjunctive kriging over ordinary kriging is that it allows estimation of functions of the primary variable, which here is the grade (Fe %) of the ore. In particular it permits estimation of indicator functions defined using thresholds on the primary variable. Miners must decide for each unsampled location whether the ore is to be sent for processing (if its concentra-

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tion exceeds the economic threshold) or to the waste dump. For this, it is necessary to determine the probability that the true value exceeds the threshold. The main advantage of disjunctive kriging over the simpler techniques is in providing these probabilities. These probabilities should enable a miner or his advisor to assess the risks associated with imprecise estimates. The probability that a critical value is exceeded depends on the distribution function. Thus the distribution function of the random variable used for thresholding needs to be estimated at specified locations. Disjunctive kriging involves transforming data to a normal distribution and then determining for each point of interest the probability that the true value exceeds the threshold.

Gaussian Disjunctive Kriging (DK) is based on an underlying “diffusion” model (where, in general, grade tends to move from lower to higher values and vice versa in a relatively continuous way). The initial data are transformed into values with a Gaussian distribution, which can easily be factorized into independent factors called Hermite polynomials as it is indicated by Rivoirard [2] for a full explanation and definition of Hermite polynomials and disjunctive kriging. In fact, any function of a Gaussian variable, including indicators, can be factorized into Hermite polynomials. These factors are then kriged separately and recombined to form the DK estimate. The major advantage of DK is that you only need to know the variogram of the Gaussian transformed values in order to perform all the krigings required.

The basic hypothesis made is that the bivariate distribution of the transformed values is bigaussian, which is testable. Although order relationships can occur, they are very small and quite rare in general. A very powerful and consistent change of support model exists for DK: the discrete Gaussian model [3].

Gaussian disjunctive kriging has been proven that it is relatively sensitive to stationary decisions, (in most cases simple kriging is used in the estimation of the polynomials). DK should thus only be applied to strictly homogeneous zones.

In this study a comparison is made between the results of ordinary kriged and disjunctive kriged grade estimates of Choghart Iron Ore Deposit. This includes a comparison of grade estimated error variance and average grade estimated by ordinary kriging and disjunctive kriging. Advantageous of disjunctive kriging over ordinary kriging is also presented. In application of disjunctive kriging, all of the statistical and geostatistical calculations and graphical output generated for this case study was made using the software system developed by the author. For the application of disjunctive kriging a series of codes in Matlab software have been used. For using ordinary kriging author has used Wingslib Software.

2. Theory

2.1. Ordinary Kriging

Ordinary kriging is a spatial interpolation estimator used to find the best linear unbiased estimate of a second-order stationary random field with an unknown constant mean as follows:

$$\hat{Z}(x_0) = \sum_{i=1}^n \lambda_i Z(x_i)$$

Where $\hat{Z}(x_0)$: kriging estimate at location x_0 ; $Z(x_i)$: sampled value at location x_i ; and λ_i : weighting factor for $Z(x_i)$. The estimation error is:

$$\hat{Z}(x_0) - Z(x_0) = R(x_0) = \sum_{i=1}^n \lambda_i Z(x_i) - Z(x_0)$$

Where $Z(x_0)$: unknown true value at x_0 ; and $R(x_0)$: estimation error. For an unbiased

estimator, the mean of the estimation error must equal zero. Therefore, $E\{R(x_0)\} = 0$

$$\text{and } \sum_{i=1}^n \lambda_i = 1.$$

The best linear unbiased estimator must have minimum variance of estimation error. The minimization of the estimation error variance under the constraint of unbiasedness leads to a set of simultaneous linear algebraic equations for the weighting factors, λ_i , which can be solved by an optimization routine and the method of Lagrange multipliers [4].

2.2. Disjunctive Kriging

Principles of disjunctive kriging of blocks and estimating reserves were given in detail by earlier workers [5]. Basically, in DK, variable Z_0 to be estimated is decomposed into a sum of disjoint (uncorrelated) components of sample values. When kriging of the separate components is possible, the procedure is feasible, i.e., when the joint probability density function of Z_0 (or the transformed Y_0) and each sample Z_a (or Y_a) is of isofactorial type.

In practice, a continuous variable like iron grade of an ore deposit can always be transformed by anamorphosis into a gaussian equivalent Y , and then only a joint Gaussian hypothesis for the probability density function (PDF) of samples and blocks is required.

We start by transforming the measured variable, $Z(x)$, to one $Y(x)$ that has a standard normal distribution such that:

$$Z(x) = \Phi[Y(x)]$$

This is done using Hermite polynomials, which are related to the normal distribution by Rodriguez's formula:

$$H_k(y) = \frac{1}{\sqrt{k!}g(y)} \frac{d^k g(y)}{dy^k}$$

in which $g(y)$ is the normal probability density function, k is the degree of the

polynomial taking values 1, 2, ... and $\frac{1}{\sqrt{k!}}$ is a

standardizing factor. The first two Hermite polynomials are:

$$H_0(y) = 1,$$

$$H_1(y) = -y;$$

Thereafter the higher order polynomials obey the recurrence relation:

$$H_k(y) = -\frac{1}{\sqrt{k}} y H_{k-1}(y) - \sqrt{\frac{k-1}{k}} H_{k-2}(y)$$

The Hermite polynomials are orthogonal with respect to the weighting function $(-y^2/2)$ on the interval from $-\infty$ to $+\infty$; they are independent components of the normal distribution of ever increasing detail. Many functions of $Y(x)$ can be represented as the sum of Hermite polynomials:

$$f\{Y(x)\} = f_0 H_0\{Y(x)\} + f_1 H_1\{Y(x)\} + f_2 H_2\{Y(x)\} + \dots,$$

Because the polynomials are orthogonal we can calculate the coefficients required for Eq. (1) as:

$$Z(x) = \Phi[Y(x)] = \Phi_0 H_0\{Y(x)\} + \Phi_1 H_1\{Y(x)\} + \Phi_2 H_2\{Y(x)\} + \dots = \sum_{k=0}^{\infty} \Phi_k H_k\{Y(x)\}$$

The transform is invertible, and so we can express the results in the original units of measurement. To kriging the variable of interest, $Z(x)$, we simply kriging the Hermite polynomials separately and sum their estimates to give the disjunctive kriging estimator:

$$\hat{Z}^{DK}(x) = \Phi_0 + \Phi_1 \hat{H}_1^k\{Y(x)\} + \Phi_2 \hat{H}_2^k\{Y(x)\} + \dots$$

Using n points in the neighborhood of estimation point x_0 where we want an estimate we estimate the Hermite polynomials by

$$\hat{H}_k^k\{Y(x_0)\} = \sum_{i=1}^n \lambda_{ik} H_k\{Y(x_i)\}$$

where the λ_{ik} are the kriging weights which are found by solving the simple kriging equations:

$$\sum_{i=1}^n \lambda_{ik} \text{cov}[H_k\{Y(x_j)\}, H_k\{Y(x_i)\}] = \text{cov}[H_k\{Y(x_j)\}, H_k\{Y(x_0)\}] ; \text{ for all } j$$

Or alternatively

$$\sum_{i=1}^n \lambda_{ik} \rho^k(x_i - x_j) = \rho^k(x_j - x_0); \text{ for all } j$$

where $\rho(x_i - x_j)$ is correlogram between points x_i and x_j .

In particular, the procedure enables us to estimate $Z(x_0)$ by

$$\hat{Z}(x_0) = \Phi\{\hat{Y}(x_0)\} = \Phi_0 + \Phi_1[\hat{H}_1^k\{Y(x_0)\}] + \Phi_2[\hat{H}_2^k\{Y(x_0)\}] + \dots$$

The kriging variance of $\hat{H}_k\{Y(x)\}$ is

$$\sigma_k^2 = 1 - \sum_{i=1}^n \lambda_{ik} \rho^k(x_i - z_0)$$

And the disjunctive kriging variance of $\hat{f}[Y(x_0)]$ is

$$\sigma_{DK}^2(x_0) = \sum_{k=1}^{\infty} f_k^2 \sigma_k^2(x_0)$$

Once the Hermite polynomials have been estimated at x_0 we can estimate the conditional probability that the true value there exceeds the critical value, z_c . The transformation $Z(x) = F[Y(x)]$ means that z_c has an equivalent y_c on the standard normal scale. Since the two scales are monotonically related their indicators are the same:

$$\Omega[Z(x) \leq z_c] = \Omega[Y(x) \leq y_c]$$

For $\Omega[Y(x) > y_c]$, which is the complement of $\Omega[Y(x) \leq y_c]$, the k th Hermite coefficient is

$$f_k = \int_{-\infty}^{+\infty} \Omega[y \leq y_c] H_k(y) g(y) dy = \int_{-\infty}^{y_c} H_k(y) g(y) dy$$

The coefficient for $k=0$ is the cumulative distribution to y_c :

$$f_0 = G(y_c)$$

And for larger k

$$f_k = \frac{1}{\sqrt{k}} H_{k-1}(y_c) g(y_c)$$

The indicator can be expressed in terms of the cumulative distribution and the Hermite polynomials:

$$\Omega[Y(x) \leq y_c] = G(y_c) + \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} H_{k-1}(y_c) g(y_c) H_k\{Y(x)\}$$

Its disjunctive kriging estimate is obtained by

$$\hat{\Omega}^{DK}[Y(x_0) \leq y_c] = G(y_c) + \sum_{k=1}^L \frac{1}{\sqrt{k}} H_{k-1}(y_c) g(y_c) \hat{H}_k^k\{Y(x_0)\}$$

where L is some small numbers.

The kriged estimates $\hat{H}_k^k\{Y(x_0)\}$ approach 0 rapidly with increasing k , and so summation need extend over only few terms. This is the same as $\hat{\Omega}^{DK}[Z(x_0) \leq z_c]$.

In this instance, we are interested in the exceedence probability as follows:

$$\hat{\Omega}^{DK}[Z(x_0) > z_c] = \hat{\Omega}^{DK}[Y(x_0) > y_c] = 1 - G(y_c) - \sum_{k=1}^L \frac{1}{\sqrt{k}} H_{k-1}(y_c) g(y_c) \hat{H}_k^k\{Y(x_0)\}.$$

3. Case Study

The Choghart iron deposit (55°28'2"E, 31°42'00"N) occurs in the Bafq mining district of Central Iran, 12 km northeast of Bafq town and 125 km southeast of Yazd city (Fig. 1).

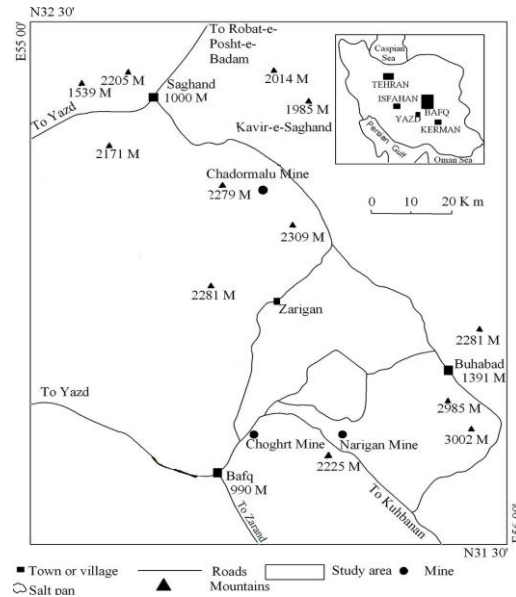


Figure 1. Geographic map showing the location of Choghart iron ore deposit in the Bafq mining district of Iran

The origin of Choghart iron deposit and other similar iron oxide deposits in the Bafq mining district, like their counterparts in the

rest of the world, has been the subject of continuing controversy for local geologists with the difference that the controversy has been fueled by the lack of absolute age determinations, accurate isotopic and fluid inclusion studies, and reliable analytical data. Some authors believe that these have been formed directly from magmas filling volcanic diatremes or flowing as lavas [6-8] while others suggest metasomatic replacement of preexisting rocks by hydrothermal solutions charged with iron leached from cooling felsic plutons [9-10].

Choghart iron ore deposit was explored by 134 boreholes (Fig. 2). The data file gives the name of each drill, the coordinate of drills, the grade of each element, measure depth, azimuth of each drill, the inclination of boreholes and level, and lithology coding etc. In general, the drilling grid is irregular (see Fig. 2). As the bench height for mining is fixed at 12.5 m, borehole samples were regularized at 12.5 m intervals.

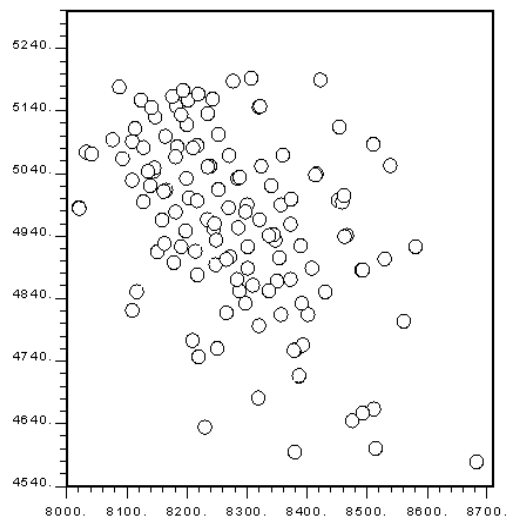


Figure 2. Borehole sample location map of Choghart iron ore deposit

The histogram of the iron concentrations (Fig. 3) is almost normal. Under these conditions, as suggested by Chile's and

Delfiner [5], a gaussian model is preferred. Summary statistics for the data set (see Table 1) exhibit the very lowly skewed nature of the distribution.

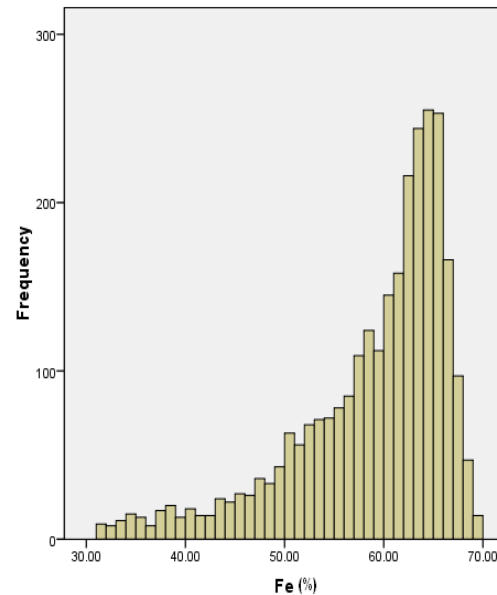


Figure 3. Histogram of borehole sample iron concentrations of Choghart deposit

Table 1. Summary statistics for the iron grades of the Bore hole data

Data set	N	Mean (%)	SD	Min. (%)	Max. (%)	Sk
Bore hole	2804	58.78	7.83	31.26	69.95	-1.303

SD – Standard deviation; Sk - Skewness

4. Variography

Variography was done in different directions. The result has been shown in Table 2. The sill and maximum range in different directions are almost the same. So there is no severe anisotropy. Therefore only an omnidirectional variogram is considered for modelling.

Figure 4 shown omnidirectional variogram and its fitted model for Choghart iron ore deposit. Model consists of a pure nugget effect with 0.405 plus a spherical scheme with sill 1 and range 270 m (Fig. 4).

Table 2. Result of variography of iron concentration for Choghart deposit in different directions

Azimuth	Dip	Sill (%) ²	Nugget effect (%) ²	Maximum range (m)
0	45	1.1	0.405	270
0	90	1	0.4	270
45	0	1.1	0.410	250
45	90	1	0.405	280
90	0	1.15	0.405	290

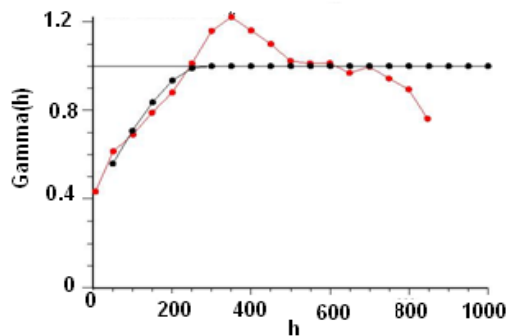


Figure 4. Omnidirectional variogram and its fitted model for Choghart iron ore deposit

5. Estimation

To verify the practical usefulness of disjunctive kriging and ordinary kriging to estimate block values, the theory of both methods have been applied to Choghart iron ore deposit. For each $20\text{ m} \times 20\text{ m} \times 12.5\text{ m}$ panel, the Fe grade was calculated by using disjunctive kriging and ordinary kriging. Figs. 5 and 6 respectively illustrate the estimated iron concentration computed by Dk and Ok for the level of 1000 m for Choghart iron ore deposit. Fig. 7 and Fig. 8 respectively show the estimated error variance maps computed by DK and OK. The estimate of the grade of the survey samples only is not enough. So we seek to forecast the probability of the Fe grade pass the threshold. One of the most important advantageous of Dk over Ok is that Dk method can provide the probability of the grade pass the threshold value. Fig. 9 shows probability map of the grade above threshold of 40 %.

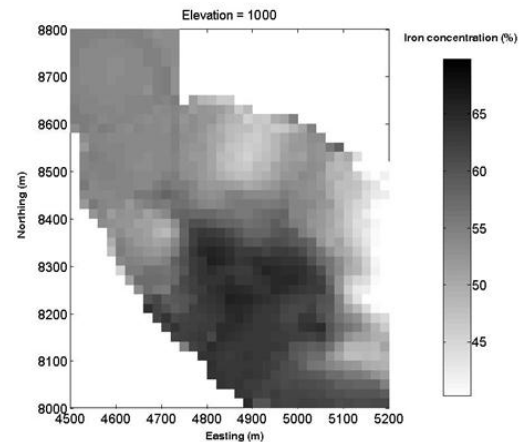


Figure 5. Estimated iron concentration computed by Dk

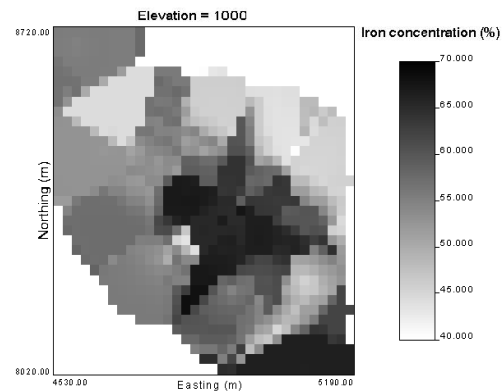


Figure 6. Estimated iron concentration computed by Ok

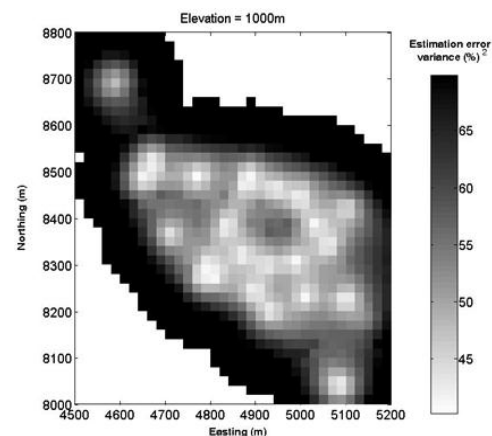


Figure 7. Estimated error variance maps computed by DK

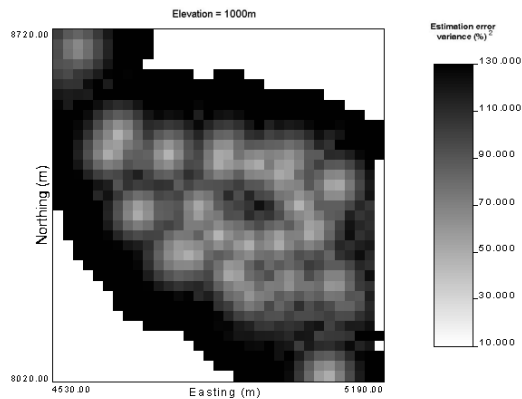


Figure 8. Estimated error variance maps computed by OK

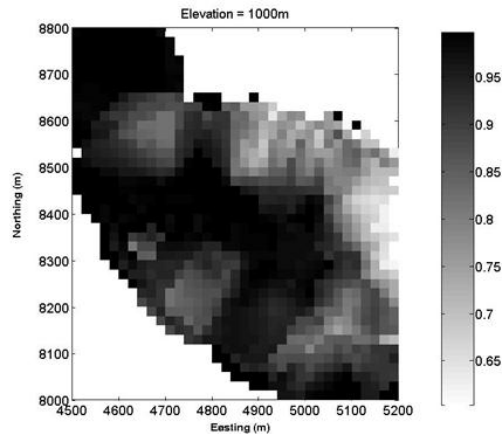


Figure 9. Probability map of the grade above threshold of 40 %

6. Conclusion

The study showed that both disjunctive kriging and ordinary kriging can be applied successfully for estimating and modeling the grade of Choghart iron ore deposit. Linear estimation of regionalized variables by ordinary Kriging results in high estimation variances, i.e. the estimates have very low precision. So evaluation of project economics based on ordinary kriging estimation is more risky. Disjunctive kriging perform better and the lower estimation variance allows less risky economic decision-making. In addition DK is able to model the uncertainty of mapping Fe

concentrations in an iron ore deposit. Ordinary kriging cannot provide such maps. In the case of Choghart iron ore deposit the average of disjunctive kriging estimation error variance is 72.076 while the average of ordinary kriging estimation error variance is 100.278 (table 3).

Table 3. Results of ordinary kriging (OK) and disjunctive kriging (DK)

	DK	OK
Mean	54.88799	55.2908
Error variance	72.07672	100.2786
Min value	31.17	33.33
Max value	67.718	68.11579

So estimation with ordinary kriging is more risky. As compared to the disjunctive kriging mean of 54.89, the mean of the ordinary kriging is 55.29, which is almost the same. However, the minimum value of the disjunctive kriging estimator is 31.17 as against 33.33 in the case of ordinary kriging. The maximum value of disjunctive kriging is 67.72, whereas that of ordinary kriging is 68.12. In DK, a conditional probability distribution is estimated for mining blocks. This means that the average grade over each mining block, of a function describing the distribution of point grades in the space, is estimated.

7. References

- [1] Vann, J, Geoval, D.G. (1998) An Overview of Non-linear Estimation. Symposium on Beyond Ordinary Kriging.
- [2] Rivoirard, J. (1994) Introduction to disjunctive kriging and non-linear geostatistics. Clarendon Press (Oxford) p. 180.
- [3] Vann, J., Sans. (1995) Global Resource Estimation and Change of Support at the Enterprise Gold Mine, Pine Creek, Northern Territory-Application of the

- Geostatistical Discrete Gaussian Model. Proceedings of APCOM XXV, Aus IMM. (Parkville), pp. 171-179.
- [4] Journel, A.G., Huijbergts, C.H.J. (1978) Mining Geostatistics, 2nd printing (Revised). Academic, Press, London.
- [5] Chilès, J.-P., Delfiner, P. (1999) Geostatistics: Modeling Spatial Uncertainty. Wiley, New York, NY.
- [6] Förster, H., Borumandi, H. (1971) Jungpräkambrische magnetit-lava und magnetit-tuffe aus dem Zentraliran. Die Naturwissenschaften, 58: 524-525.
- [7] Förster, H., Jafarzadeh, A. (1994) The Bafq mining district in Central Iran: A highly mineralized Infracambrian volcanic field. Economic Geology, 89: 1697-1721.
- [8] Samani, B.A., (1988) Metallogeny of the Precambrian in Iran. Precambrian Research, 39: 85-106.
- [9] Daliran, F. (1999) REE geochemistry of Bafq apatites, Iran; implication for the genesis of Kiruna-type iron ores. In: Stancly et al. (Eds.) Mineral Deposits; Processes to Processing. Balkema, Rotterdam, pp. 631-634.
- [10] Gerasimovsky, V.V., Minjeev, D.A. (1981) Redkozemelyne elementy v apatitach izapatit-magnetiovykh. Rud Geochimija, 1: 99-104.